ON SOME EXTREMAL PROBLEMS
FOR DIFFERENTIABLE FUNCTIONS OF ONE VARIABLE

UDC 517.5

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ABSTRACT. The duality of the problem concerning the least constant in
the Hadamard-Kolmogorov inequality with the problem concerning the best approx-
imation of a class of differentiable functions by another similar class of smoother
functions as well as the duality of the Stečkin problem concerning the best approx-
imation of a differentiation operator by bounded linear operators with the prob-
lem concerning the linear approximation of a class by a class are proved in the $L_p$
spaces on the real line and on the half-line $[0, \infty)$.

Bibliography: 28 titles.

Introduction

In the present paper we state dual problems for sharp Hadamard-Kolmogorov in-
equalities involving the norms of functions and their derivatives and for the Stečkin
problem concerning the best approximation of differentiation operators by bounded
operators.

In the Introduction the problems are posed, a survey of the known results is
given and some simple properties of the quantities under consideration are cited.

In the sequel $k$ and $n$ are integers such that $0 \leq k < n$, the letters $p, q$ and $r$
denote parameters satisfying the condition $1 \leq p, q, r \leq \infty$, and $I$ is either the real line
or the half-line $[0, \infty)$. The notation $\|x\|_p$ will be used for the norm of a function $x$
in the space $L_p = L_p(I)$, i.e. we put

$$
\|x\|_p = \|x\|_{L_p(I)} = \left\{ \left( \int_I |x(t)|^p \, dt \right)^{1/p}, \quad 1 \leq p < \infty, \right.
\left. \text{ess sup}_{t \in I} |x(t)|, \quad p = \infty. \right\}
$$

Let $W^n_p$ for $1 < p \leq \infty$ be the set of functions $x$ whose derivative $x^{(n-1)}$ of
order $n-1$ is locally absolutely continuous on $I$ and whose nth derivative $x^{(n)} \in L_p$,
and let $W^n_1$ be the set of functions $x$ of the form

$$
x(t) = \sum_{i=0}^{n-1} \frac{a_i}{i!} t^i + \frac{1}{(n-1)!} \int_0^t (t - \tau)^{n-1} d\eta(\tau),
$$

where the $a_i$ are real numbers, $\eta$ is a function of bounded variation and $\eta(0) = -a_{n-1}$
(and hence $x^{(n-1)} = \eta$); when $x \in W^n_1$, we will write $\|x^{(n)}\|_1$ in place of $\sqrt{x^{(n-1)}}$.

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Finally, we denote by $W^n_{p,r}$ the intersection of the spaces $L_r$ and $W^n_p$, and by $Q = Q^n_{p,r}$ the set of functions $x \in W^n_{p,r}$ having the property $\|x^{(n)}\|_p \leq 1$.

We will consider the following quantity for an arbitrary pair of numbers $\mu_0 > 0$, $\mu_n > 0$:

$$\mu_k = \sup \|x^{(k)}\|_q,$$

where the supremum is taken over all functions $x \in W^n_{p,r}$ satisfying the conditions $\|x\|_p \leq \mu_0$, $\|x^{(n)}\|_p \leq \mu_n$. By making use of the homogeneity of the norm in $L_s(I)$ under the substitution of $c x(ht)$ for $x(t)$, one can obtain the dependence of $\mu_k$ on $\mu_0$ and $\mu_n$; namely, the following formula holds (see [1]):

$$\mu_k = K \mu_0^\alpha \mu_n^\beta,$$

in which

$$\alpha = \frac{n-k-p-1}{n-p-1+q^{-1}}, \quad \beta = 1 - \alpha,$$

and $K$ is a constant not depending on $\mu_0$ or $\mu_n$.

It follows that the functions $x \in W^n_{p,r}$ satisfy the inequality

$$\|x^{(k)}\|_q \leq K \|x\|_p \|x^{(n)}\|_p^{\beta} .$$

(0.1)

Such inequalities were first considered by Hardy and Littlewood [2], Landau [3] and Hadamard [4]. Important results in the problem of investigating inequality (0.1) have been obtained by Kolmogorov [5] and Sz.-Nagy [1]. At the present time the constant $K$ and the set of extremal functions on which (0.1) becomes an equality are known for many values of the parameters $k, n, p, q, r$ and $I$ (see the bibliographies in [6]–[8]). Definitive conditions for the finiteness of the constant $K$ in (0.1) have been obtained by Gabušin [6]. He proved that $K$ is finite if and only if

$$\frac{n-k}{r} + \frac{k}{p} \geq \frac{n}{q} .$$

(0.2)

We will also consider the Stečkin problem [9], [10] concerning the best approximation

$$E(N) = \inf_{\|T\|_{L_1,L_q} \leq N} \sup_{x \in Q^n_{p,r}} \|x^{(k)} - Tx\|_q$$

(0.3)

of the differentiation operator of order $k$ by bounded linear operators $T$ from $L_r$ into $L_q$ on the class of $n$-times differentiable functions. The investigation of the quantity (0.3) includes a study of the existence, uniqueness and other properties of an extremal operator, i.e. an operator $T$ satisfying the conditions $\|T\| = \|T\|_{L_1,L_q} \leq N$ and $E(N) = U(T)$, where

$$U(T) = \sup_{x \in Q} \|x^{(k)} - Tx\|_q .$$

If $x \in W^n_{p,r}$, then $x^{(k)}$ is a continuous function unless $k = n - 1$ and $p = 1$. It is assumed in this connection that if $q$ or $r$ is infinite, then $L_q$ or respectively $L_r$ is the space $C$ of continuous functions (unless $q = \infty$, $k = n - 1$, $p = 1$).
Stečkin [10] noted a connection between the quantity $E(N)$ and the least constant $K$ in (0.1). This connection is expressed by the inequality

$$E(N) \geq \Omega(N),$$

(0.4)

where

$$\Omega(N) := \sup_{x \in Q} (\|x^{(k)}\|_{L^r} - N \|x\|_{L^r}) = \beta x^{2/3} K^{1/3} N^{-x/3}.$$

(0.5)

The following formula holds for $E(N)$:

$$E(N) = N^{-\gamma} E(1), \quad \gamma = \frac{x}{3} = \frac{n - k - p^{-1} + q^{-1}}{k + r^{-1} - q^{-1}};$$

it is proved in [10] for $p = q = r$, while the general case, which is obtainable by the same method, is cited in [11]. A solution of problem (0.3) for concrete values of the parameters is given in [7]–[10] and [12]–[18]. In some of these cases the corresponding sharp inequality (0.1) was previously known, while in others (see [14], [15], [17] and [18]) the solution permitted one to find the least constant $K$ in (0.1). We note that a solution of problem (0.3) can be obtained in the case $p = q = r = \infty, I = (-\infty, \infty)$ with the use of a result of Domar [19] (see [10] for $n = 2, 3$, and [12] for $n = 4, 5$).

The quantity $E(N)$ is not always finite; as Gabušin has shown [20], it is finite if and only if

$$q \gg p, \quad q \gg r.$$  

(0.6)

For $q = \infty$ it is natural to consider in conjunction with $E(N)$ the quantity

$$e(N) = \inf_{||T|| \leq N} \sup_{x \in Q, \partial Q} (x^{(k)}(0) - T x)$$

(0.7)

of best approximation of the value of a differentiation operator at $t = 0$ by bounded linear functionals $T$ (see [9]–[12], [14]–[16] and others). Gabušin [16] proved that the equality

$$e(N) := \sup_{x \in Q} (x^{(k)}(0) - N \|x\|_{L^r}) = \beta x^{2/3} K^{1/3} N^{-x/3}$$

(0.8)

holds (here $q = \infty$). But $e(N)$ coincides with $E(N)$ for $q = \infty$ (see [11] as well as [9], [10], [12], [14]–[16], [18] and Corollary 2 below); therefore inequality (0.4) becomes an equality when $q = \infty$. The equality $E(N) = \Omega(N)$ also holds in some other cases, e.g. (see [13]) when $p = q = r = 2$ and $I = (-\infty, \infty)$.

The constant $K$ in (0.1) and the quantity (0.3) are connected with the approximation of one class of functions by another, i.e. with a quantity of the form

$$\sup_{\Phi} \inf_{\Psi} \|\psi - \varphi\|_{L^r}.$$ 

The latter problem arose as a subsidiary problem in, for example, [21] and [22]. It was studied as an independent problem by Taǐkov [23], who indicated a connection between it and a problem of form (0.3). It will be discussed here for two pairs of classes of differentiable functions.
Let $1 \leq p', q', r' \leq \infty$ and let $m, n$ be integers such that $0 < m \leq n$, let $B_{r'}^m(N) = B_{r'}^m(N, I)$ be the set of functions $\phi \in W_{r'}^m$ for which $\|\phi^{(m)}\|_{r'} \leq N$, let $B_{q'}^n = B_{q'}^n(1, I)$ and let $\mathfrak{N}(N) = \mathfrak{N}(N, I)$ be the set of linear (homogeneous and additive) operators $S$ from $W_{q'}^m$ into $W_{r'}^m$ such that

$$\|(S\phi)^{(m)}\|_{r'} \leq N \|\phi^{(m)}\|_{q'}$$

(0.9)

for $\phi \in W_{q'}^m$. We put

$$F(\Psi, N) = F(\Psi, N, I) = \inf_{\varphi \in B_{r'}^m(N)} \|\Psi - \varphi\|_{p'}, \quad \Psi \in W_{r'}^m,$$

$$J(S) = J(S, I) = \sup_{\psi \in B_{q'}^n} \|\psi - S\psi\|_{p'}, \quad S \in \mathfrak{N}(N).$$

Then the quantity

$$F(N) = F(N, I) = \sup_{\psi \in B_{q'}^n} F(\psi, N, I) \tag{0.10}$$

is the approximation in the metric of $L_{p'}$ of the class $B_{q'}^m$ by the class $B_{r'}^m(N)$ while

$$G(N) = G(N, I) = \inf_{S \in \mathfrak{N}(N)} J(S) \tag{0.11}$$

is the corresponding linear approximation of a class by a class. Clearly,

$$F(N) \leq G(N). \tag{0.12}$$

The quantities $F(N)$ and $G(N)$ have been studied in [24], [13], [25] and [26]. In particular, it is known [25] that

$$F(N) = N^{-\gamma} F(1), \quad G(N) = N^{-\gamma} G(1), \tag{0.13}$$

$$\gamma = \left(m + \frac{1}{p'} - \frac{1}{q'} \right) \left(n - m + \frac{1}{q'} - \frac{1}{r'} \right),$$

and that the quantity $F(N)$ is finite if [26] and only if [25]

$$\frac{m}{r'} + \frac{n - m}{p'} \leq \frac{n}{q'} \tag{0.14}$$

We cite values of the parameters $p', q', r'$ and $n$ for which the quantity $F(N)$ is known (Table 1).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$q'$</th>
<th>$r'$</th>
<th>$p'$</th>
<th>Authors</th>
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<td>2</td>
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<td>$\infty$</td>
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<td>Subbotin [24], Arestov, Gabušin [25]</td>
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</table>

We will consider similar problems for another pair of classes. Let $w_{r'}^n$ and $b_{r'}^n(N)$ $= b_{r'}^n(N, I)$ be respectively the sets of functions $x \in W_{r'}^n$ and $x \in B_{r'}^n(N)$ satisfying the condition $x^{(i)}(0) = 0$ for $i = 0, 1, \ldots, n - 1$; in exactly the same way, let $w_{q'}^m$ and
$b_q^m$ be the sets of functions $x \in W_q^m$ and $x \in B_q^m$ with the property $x^{(i)}(0) = 0$ for $i = 0, 1, \ldots, m - 1$. We put

$$f(\Psi, N) = f(\Psi, N, I) = \inf_{\varphi \in b_q^m} \|\Psi - \varphi\|_{p'},$$

$$f(N) = f(N, I) = \sup_{\varphi \in b_q^m} f(\Psi, N, I). \quad (0.15)$$

In addition, we denote by $\mathfrak{N}'(N) = \mathfrak{N}'(N, I)$ the set of linear operators $S$ from $w_q^m$ into $w_p^m$ satisfying (0.9), and put

$$j(S) = j(S, I) = \sup_{\varphi \in b_q^m} \|\Psi - S\varphi\|_{p'},$$

$$g(N) = g(N, I) = \inf_{S \in \mathfrak{N}'(N)} j(S). \quad (0.16)$$

It is not difficult to show that formulas (0.13) with $F$ and $G$ replaced by $f$ and $g$ hold for (0.15), (0.16) (see the proof of (0.13) in [25]). By virtue of the equality

$$F(N) = \sup_{\varphi \in b_q^m} \inf_{\varphi \in b_q^m} \|\Psi - \varphi\|_{p'},$$

and the imbedding $B_p^m(N) \subset B_r^m(N)$ we have

$$F(N) \leq f(N). \quad (0.17)$$

Also

$$G(N) \leq g(N), \quad (0.18)$$

since the formula

$$S\varphi = \varphi - \varphi_0 + S'\varphi_0,$$

where

$$\varphi_0(t) = \Psi(t) - \sum_{i=0}^{n-1} \Psi^{(i)}(0) \frac{t^i}{i!}, \quad \Psi \in W_q^m,$$

associates with each operator $S' \in \mathfrak{N}'(N)$ an operator $S \in \mathfrak{M}(N)$ for which $f(S') = f(S)$.

Sometimes $G(N)$ (and hence $F(N)$) is bounded from above by $E(N)$ (see [23], [24], [13] and [25]). More precisely, suppose

$$m = n - k, \quad p = r = q', \quad p' = r' = q, \quad (0.19)$$

problem (0.3) has an extremal operator $T$ that is defined, linear and permutable with the differentiation operator of order $n$ on $W_p^m$ and (cf. (0.3))

$$\sup_{x \in H_p^m} \|x^{(k)} - Tx\|_q = E(N). \quad (0.20)$$

Then

$$G(N) \leq E(N), \quad (0.21)$$
and hence
\[ F(N) \leq E(N). \tag{0.22} \]

In fact, with each function \( \psi \in W^{n-k}_p \) we associate a function \( x \in W^n_p \) such that \( x^{(k)} = \psi \), and define an operator \( S \) by the formula
\[ S\psi = Tx, \quad \psi \in W^{n-k}_p. \]

We have
\[ (S\psi)^{(n)} = (Tx)^{(n)} = T x^{(n)} = T \psi^{(n-k)}, \]
so that condition (0.9) is satisfied for \( S \) and, consequently, \( S \in \mathcal{M}(N) \). Further, by virtue of (0.20)
\[ \| \psi - S\psi \|_q = \| x^{(k)} - Tx \|_q \leq E(N) \| x^{(n)} \|_p = E(N) \| \psi^{(n-k)} \|_p, \]
which implies (0.21).

In the present paper we establish a connection between problem (0.1) and problems (0.10), (0.15) and between problem (0.3) and problems (0.11), (0.16) under the following relations between the parameters:
\[ m = n - k, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad \frac{1}{q} + \frac{1}{q'} = 1, \quad \frac{1}{r} + \frac{1}{r'} = 1. \tag{0.23} \]

It is shown in particular (Theorem 2) that the following equalities hold under the fulfillment of conditions (0.23):
\[ \Omega(N) = \beta x^{\alpha/\beta} K^{1/\beta} N^{-\alpha/\beta} = \begin{cases} F(N), & I = (-\infty, \infty), \\ f(N), & I = [0, \infty), \end{cases} \tag{0.24} \]

i.e. problems (0.10) and (0.15) are dual to (0.1) when \( I = (-\infty, \infty) \) and \( I = [0, \infty) \) respectively. In addition, we prove (Theorem 4) that the quantities \( F(N) \) and \( f(N) \) do not depend on the set \( I \). These two facts permit us to state the values of \( F(N) \) and \( f(N) \) in those cases when a sharp inequality (0.1) is known. Many of these cases for \( F(N) \) are collected in Table 2; in the last column we indicate the authors of inequality (0.1) for the corresponding values of the parameters (when \( I = (-\infty, \infty) \)).

<table>
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<tr>
<th>( n )</th>
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<td>Taikov [14]</td>
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<td>( \frac{r'}{r' - 1} )</td>
<td>Hardy, Littlewood, Pólya [27]</td>
</tr>
</tbody>
</table>

We note that the inequality
\[ F(N) \leq E(N) \tag{0.25} \]
follows from (0.4), (0.24) and (0.17) under conditions (0.23), which differ from conditions (0.19) in (0.22).

In Theorem 3 we assert that the equalities
\[
E(N) = \begin{cases} 
G(N), & \text{if } I = (-\infty, \infty), \\
g(N), & \text{if } I = [0, \infty);
\end{cases}
\]  
(0.26)
are valid for \(1 < q \leq \infty\), \(1 < p \leq \infty\); here, as before, the parameters are subject to conditions (0.23).

Theorems 2 and 3 are concretizations of results of the note [28].

If \(k = 0\) and \(q = r\), then, clearly, \(K = 1\) while \(E(N) = 0\) for \(N \geq 1\) and \(E(N) = \infty\) for \(N < 1\). In exactly the same way, if \(m = n\) and \(q' = r'\), then \(F(N) = G(N) = 0\) for \(N \geq 1\) and \(F(N) = G(N) = \infty\) for \(N < 1\); an analogous remark can be made for \(f(N), g(N)\).

Thus, when \(k = 0\) and \(q = r\), i.e. (see (0.23)) when \(m = n\) and \(q' = r'\), equalities (0.26) hold while equalities (0.24) are degenerate (\(\beta = 0\)). Therefore the conditions
\[
\frac{n-k}{r} + \frac{k}{p} \geq \frac{n}{q}, \quad \text{with} \quad q \gg r, \quad \text{if} \quad k = 0,
\]  
(0.27)
will sometimes be used in place of (0.2).

§1. Auxiliary assertions

Let \(W^t_s\) denote the space \(W^t_s\) if \(I = (-\infty, \infty)\), and the space \(w^t_s\) if \(I = [0, \infty)\); the classes \(B^t_s(N), B^t_s\) and quantities \(I(\psi, N), I(N), J(S), G(N)\) are defined analogously.

Finally, we associate with an integer \(i \in [0, n]\) and a function \(\chi \in W^t_{s-i}\) the functional
\[
M^t_{\chi} = \begin{cases} 
(-1)^{n-i} \int x^{(i)}(t) \chi^{(n-i)}(t) \, dt, & 1 \leq s \leq \infty, \\
(-1)^{n-i} \int x^{(i)}(t) \, d\chi^{(n-i)}(t), & s = 1;
\end{cases}
\]  
(1.1)
we put \(M^0_{\chi} = M_{\chi}\) when \(i = 0\). Here and below the symbol \(I\) is omitted in integrals over \(I\).

**Lemma 1.** Suppose conditions (0.27) and (0.23) are satisfied and the difference \(\omega = \psi - \phi\) of a pair of functions \(\psi \in W^p_{n-k}, \psi \in W^p_{r'}\) belongs to \(L_p\), the function \(\psi\) being continuous outside some finite interval if \(p = 1, q = \infty\) and \(k = n - 1\).

Then for any function \(x \in W^p_{n,r}\)
\[
M^p_{\omega}x - M^p_x = \begin{cases} 
x^{(n)}(t) \omega(t) \, dt, & 1 < p \leq \infty, \\
\omega(t) \, dx^{(n-1)}(t), & p = 1.
\end{cases}
\]  
(1.2)

**Proof.** Let \(k\) be an infinitely differentiable function on \(I\) such that \(0 \leq k \leq 1, k(t) = 1\) for \(|t| \leq 1\) and \(k(t) = 0\) for \(|t| \geq 2\). We put \(x_\delta(t) = k(t\delta^{-1}) \psi(t)\) (\(\delta > 0\)).

Clearly, the function \(x_\delta\) has compact support and belongs to \(W^p_{n,r}\) (if \(x \in W^p_{n,r}\)).

We obtain (1.2) for \(x_\delta\) by integrating the integrals in its right sides by parts and taking into account the fact that \(\varphi \in W^p_{r}\) and \(\psi \in W^p_{r-k}\). We will show that its validity for \(x\) follows in the limit for \(\delta \to \infty\).

The relation \(M^p_{\varphi}x_\delta \to M^p_x\) is obvious.

In the case \(1 < p \leq \infty\) we have, according to Leibniz' formula,
\[ x^{(n)}_\delta(t) = x^{(n)}(t) \kappa(t\delta^{-1}) + \rho(t), \]

where

\[ \rho(t) = \sum_{i=1}^{n} C_i^i x^{(n-i)}(t) \delta^{-i} \kappa^{(i)}(t\delta^{-1}). \]

The derivatives \( x^{(i)} \) for \( i = 0, 1, \ldots, n-1 \) are bounded for \( q = \infty \) by virtue of the finiteness of the constant \( K \) in (0.1), so that

\[ \| \rho \|_C \leq c_1 \delta^{-1}. \]

And since \( \rho \) vanishes outside the interval \( I = [-\delta, \delta] \),

\[ \| \rho \|_p \leq c_2 \delta^{(1/p)-1}, \quad 1 < p \leq \infty. \]

Hence

\[ \int x^{(n)}_\delta(t) \omega(t) \, dt \rightharpoonup \int x^{(n)}(t) \omega(t) \, dt \quad (\delta \to \infty) \]

for \( 1 < p \leq \infty \).

Suppose \( p = 1 \). We note at once that the case \( r = \infty, n = 1 \) is excluded by condition (0.27), since then \( k = 0 \) and, according to (0.2), \( q = \infty \), i.e. \( k = 0, q = r \).

Inasmuch as \( \sqrt[n]{x^{(n-1)}} < \infty \), the derivative \( x^{(n-1)} \) has limits for \( t \to \pm \infty \), and since \( x \in L_r \), these limits are equal to zero. As before,

\[ x^{(n-1)}_\delta(t) = x^{(n-1)}(t) \kappa(t\delta^{-1}) + \rho_1(t), \]

the function \( \rho_1 \) being absolutely continuous and such that

\[ \sqrt[n]{\rho_1} = \| \rho_1 \|_{L_r} \leq c_3 \| x^{(n-1)} \|_{L_{\infty}(I_\delta)} + c_4 \delta^{-1}, \]

where \( I_\delta \) is the set of points \( t \in I \) for which \( |t| \geq \delta \). Hence \( \sqrt[n]{\rho_1} \to 0 \) as \( \delta \to \infty \).

Since the function \( \omega = \psi - \phi \) is continuous (when \( q = \infty, p = 1 \) and \( k = n-1 \) on \( I_\delta \) for sufficiently large \( \delta \)), it follows by virtue of Helly’s theorem that

\[ \int \omega(t) \, dx^{(n-1)}_\delta(t) \to \int \omega(t) \, dx^{(n-1)}(t) \quad (\delta \to \infty). \]

The relation

\[ M^n_\delta x_\delta \rightharpoonup M^n_\delta x \quad (\delta \to \infty) \]

is verified analogously. The lemma is proved.

Let us consider the quantity

\[ u(\psi, T) = \sup_{x \in Q^n_{p,r}} \{ M^n_\delta x : T x \} \quad (1.3) \]

for a function \( \psi \in W^{n-k}_q \) and a linear functional \( T \in L^*_r \).

When \( 1 \leq r < \infty \), the functional \( T \) has the form

\[ T x = \int x(t) \lambda(t) \, dt, \quad (1.4) \]

where \( \lambda \in L_r \) and \( \| T \| = \| \lambda \|_r \); if, on the other hand, \( r = \infty \), then, at least on the set
of continuous functions having a zero limit at infinity, we have

\[ Tx = \int x(t) \, d\Lambda(t), \quad (1.5) \]

where \( \sqrt{\Lambda} = \|T\|_{C_0^*} \leq \|T\|_{L_{\infty}^c}. \)

In the case \( r = \infty \) we denote by \( T_0 \) the functional that is defined by (1.5) on the space \( C \) of continuous functions and linearly extended with preservation of norm onto the set \( L_\infty \); clearly,

\[ \|T_0\|_{L_\infty^c} = \|T_0\|_{C^*} = \sqrt{\Lambda} \leq \|T\|_{L_\infty^c}. \]

When \( 1 \leq r < \infty \) we put \( T_0 = T. \)

**Lemma 2.** Suppose conditions (0.23) and (0.27) are satisfied. If a functional \( T \in L_r^c \) and a function \( \psi \in W_q^{n-r-k} \) are such that

\[ u(\psi, T) < \infty, \]

the function \( \psi \) being continuous outside some finite interval for \( p = 1, q = \infty \) and \( k = n - 1 \), then there exists a unique function

\[ \varphi \in W_p^n \]

with the properties

\[ (-1)^{nq} \varphi^{(n)} = \lambda \quad \text{or} \quad (-1)^{nq} \varphi^{(n-1)} = \Lambda, \quad (1.7) \]

\[ \|\psi - \varphi\|_{L^p} = u(\psi, T_0) \leq u(\psi, T), \quad (1.8) \]

and the representation

\[ M_\psi^k \varphi - T_0 \varphi = \begin{cases} \int (\psi(t) - \varphi(t)) \, x^{(n)}(t) \, dt, & 1 \leq p < \infty, \\ \int (\psi(t) - \varphi(t)) \, dx^{(n-1)}(t), & p = 1 \end{cases} \quad (1.9) \]

holds for any function \( x \in W_p^n \).

**Proof.** Let \( H \) denote the set of infinitely differentiable functions with compact support, let \( H_p^n \) denote the set of functions \( x \in H \) with \( \|x^{(n)}\|_p \leq 1 \) and let

\[ v(T) = \sup_{x \in H_p^n} \{ M_\psi^k x - Tx \}; \]

clearly

\[ v(T) = v(T_0) \leq u(\psi, T). \quad (1.10) \]

A function \( x \in H \) is uniquely determined by its \( n \)th derivative, so that

\[ M_\psi^k x - Tx = Sx^{(n)}, \quad (1.11) \]

where \( S \) is a functional on the set \( Y \) of \( n \)th derivatives \( x^{(n)} \) of the functions \( x \in H \). We will regard \( Y \) as a (normed) subspace of the space \( L_p \) for \( 1 \leq p < \infty \) and of the space \( C_0 \) for \( p = \infty \). It is not difficult to verify that \( S \) is linear on \( Y \) and \( ||S||_{Y^*} = v(T). \)
Extending $S$ linearly with preservation of norm from $Y$ onto $L_p$ or $C_0$, we get

$$Sy = \int_0^T y(t) \omega(t) \, dt, \quad 1 \leq p < \infty,$$

where

$$v(T) = \left\| \omega \right\|_{L^p} \quad \text{or} \quad v(T) = \sqrt{v}.$$  \hspace{1cm} (1.12)

Integrating the integral in (1.4) or (1.5) by parts, we get

$$Tx = T_0x = \int_0^T x^{(n)}(t) \psi(t) \, dt, \quad x \in H,$$  \hspace{1cm} (1.13)

where $\psi$ is a function having properties (1.6) and (1.7). In exactly the same way, we find (on $H$) that

$$M^{k}x = \int_0^T x^{(n)}(t) \psi(t) \, dt.$$  

When $p < \infty$, (1.11) takes the form

$$\int_0^T \{\psi(t) - \psi(t)\} x^{(n)}(t) \, dt = \int_0^T \omega(t) x^{(n)}(t) \, dt, \quad x \in H.$$

It follows that $\omega = \psi - \bar{\psi} - P_0$, where $P_0$ identically vanishes when $I = [0, \infty)$, and is a polynomial of degree $n - 1$ when $I = (-\infty, \infty)$. The function $\varphi = \psi + P_0$ also yields the representation (1.13) of $T$ and has properties (1.6) and (1.7).

For $p = \infty$ we have

$$\int_0^T \{\psi(t) - \psi(t)\} x^{(n)}(t) \, dt = \int_0^T x^{(n)}(t) \, dt, \quad x \in H,$$

or

$$\int_0^T \{\delta(t) - v(t)\} x^{(n+1)}(t) \, dt = 0, \quad x \in H,$$

where

$$\delta(t) = \int_0^t \{\psi(\tau) - \bar{\psi}(\tau)\} \, d\tau + v(0).$$

And this means that $\nu = \delta - P_1$, where $P_1$ identically vanishes if $I = [0, \infty)$, and is a polynomial of degree $n$ if $I = (-\infty, \infty)$. Consequently, the function $\nu$ is absolutely continuous, its derivative $\omega = \nu' = \psi - \bar{\psi} - P_1'$ belongs to $L_1$, and $\|\omega\|_1 = \sqrt{v}$. In (1.13) the function $\varphi = \bar{\psi} + P_1'$ can be taken in place of $\bar{\psi}$.

If $\varphi$ is actually chosen in this way, then, clearly,

$$\left\| \omega \right\|_{L^p} = \left\| \psi - \bar{\psi} \right\|_{L^p} = v(T_0) - v(T) \leq u(\psi, T) < \infty.$$  

According to (1.7) and (1.1), $M_\varphi = T_0$. In addition, the conditions of Lemma 1 on the parameters and the functions $\psi$ and $\varphi$ are satisfied. Therefore (1.9) holds for any function $x \in W^{n}_{p,r}$. Hence

$$u(\psi, T_0) \leq \|\psi - \bar{\psi}\|_{L^p} = v(T_0),$$

which implies (1.8) by virtue of (1.10).
If \( I = [0, \infty) \), then the uniqueness of \( \varphi \) follows from (1.7) and (1.6) (in this case \( \varphi^{(i)}(0) = 0 \) for \( i = 0, 1, \ldots, n-1 \)). Suppose \( I = (-\infty, \infty) \) and \( \varphi_1 \) also satisfies conditions (1.6)—(1.9). By virtue of (1.7), \( \varphi_1 \) differs from \( \varphi \) by a polynomial \( P \) of degree \( n-1 \). According to (1.8), we have \( \|P\|_{p'} = \|\varphi_1 - \varphi\|_{p'} < \infty \). It follows for \( p' < \infty \) that \( P \equiv 0 \) and hence \( \varphi_1 = \varphi \). If, on the other hand, \( p' = \infty \), then \( P = c = \text{const.} \). Clearly, the function \( \varphi_1 = \varphi + c \) satisfies (1.6) and (1.7) for any constant \( c \). It also satisfies (1.9), since \( \lim_{t \to \pm \infty} x^{(n-1)}(t) = 0 \) for \( x \in W^{n,1}_{1,p} \), therefore

\[
\|\varphi_1 - \varphi - c\|_{\infty}.
\]

The equality

\[
\text{ess sup}_{t \in (-\infty, \infty)} \{\varphi_1(t) - \varphi(t)\} = \text{ess inf}_{t \in (-\infty, \infty)} \{\varphi(t) - \varphi(t)\},
\]

follows from (1.8), since we could otherwise choose the constant \( c \) so that

\[
\|\varphi_1 - \varphi - c\|_{\infty} < \|\varphi - \varphi\|_{\infty} = u(\varphi, T_0).
\]

Consequently, the function \( \varphi \) is also unique for \( p' = \infty \). Lemma 2 is proved.

For a fixed function \( \psi \in W^{n-1,k}_{1,q} \) we put

\[
\sigma(\psi, N) = \inf_{\|T\| \leq N} u(\psi, T), \tag{1.14}
\]

where \( \|T\| \) is the norm of a functional \( T \in L^{p'}_r \) and \( u(\psi, T) \) is defined by (1.3) and (1.1).

**Corollary 1.** Suppose conditions (0.23) and (0.27) are satisfied and \( \psi \) is a member of \( W^{n-1,k}_{q} \) that is continuous outside some finite interval when \( q = \infty \), \( p = 1 \) and \( k = n - 1 \). Then

\[
\sigma(\psi, N) = \inf_{\varphi \in B^{n-1,k}_{1,q}(N)} \|\varphi - \psi\|_{p'} \tag{1.15}
\]

In fact, the inequality \( \sigma(\psi, N) \leq F(\psi, N) \) follows from Lemma 1, since the functional \( T_\varphi \) on \( W^n_{1,p} \) by the formula \( T_\varphi x = M_\varphi x \) has a natural extension onto \( L_r \) such that \( \|T_\varphi\|_L^{p'} = \|\varphi^{(n)}\|_{r'}. \) The reverse inequality follows from Lemma 2.

**Remark 1.** As can be seen from the proofs of Lemmas 1 and 2 and Corollary 1, in addition to (1.15) it is possible to assert that to each extremal functional of problem (1.14) there corresponds a function \( \varphi \) for which \( \|\varphi - \varphi\|_{p'} = F(\psi, N) \) and, conversely, each such function induces an extremal functional.

**§2. Connection between problems in the case \( q = \infty \)**

We first consider some properties of the quantity \( e(N) \) and an extremal functional of problem (0.7).

Let \( \theta \) be the function defined on \( I \) by the relation

\[
\theta(t) = \begin{cases} \frac{(-1)^{n-k}}{(n-k-1)!} t^{n-k-1}, & \text{for } t > 0, \\ 0 & \text{for } t \leq 0; \end{cases} \tag{2.1}
\]

Clearly \( \theta \in B^{n-k}_{1}, \theta \in B^{n-k}_{1} \). We have \( M^{k}_{\delta} x = x^{(k)}(0) \), so that
\( e(N) = \sigma(\theta, N) \),

and hence, by virtue of Corollary 1,

\[
e(N) = \inf_{\varphi \in B^\infty_r(N)} \| \theta - \varphi \|_{p'}.
\]

(2.2)

Here the case \( k = 0, r = \infty \) is generally excluded. But it is not difficult to verify directly that (2.2) is also valid in this case.

There exists an extremal functional \( T \) in problem (0.7) (see [11] and [16]). By virtue of Remark 1, this means that there exists a function \( \varphi \in B^\infty_r(N) \) on which the infimum in (2.2) is achieved. Suppose \( 1 < p, r < \infty (q = \infty) \). Then \( T \) has the form

\[
Tx = \int x(t) \lambda(t) \, dt
\]

(2.3)

and, by virtue of Lemma 2, for any function \( x \in W^m_{p,r} \)

\[
x^{(k)}(0) - Tx = \int \omega(t) x^{(m)}(t) \, dt,
\]

(2.4)

where

\[
\| \omega \|_{p'} = e(N), \quad \omega = \theta - \varphi, \quad \varphi \in B^\infty_r(N),
\]

(2.5)

the function \( \theta \) is defined in (2.1), and \( \lambda = (-1)^n \varphi^{(n)} \).

According to the results of [16] (Theorems 1 and 2), there exists a function \( y \in W^m_{p,r} \) such that \( Ty = \| T \| \cdot \| y \|_{p,r}, \quad y^{(k)}(0) - Ty = e(N) \| y^{(n)} \|_p \) and \( y \) is an extremal in (0.1) \((q = \infty)\). Hence

\[
y = c_1 \| \lambda \|^{1/(r-1)} \text{sgn} \lambda,
\]

(2.6)

\[
y^{(n)} = c_2 \| \omega \|^{1/(p-1)} \text{sgn} \omega,
\]

(2.7)

where \( c_1 \) and \( c_2 \) are positive constants (relation (2.6) is given in [16]).

Relations (2.5)–(2.7) imply the following properties of \( \omega \):

1) The function \( \omega \) has continuous derivatives up to order \( n \) on the half-lines \([0, \infty)\) and \((-\infty, 0]\) (if \( I = (-\infty, \infty) \)).

2) The conditions

\[
\omega^{(i)}(0) = \begin{cases} (-1)^{n-k}, & \text{if } i = n - k - 1, \\
0, & \text{if } i \neq n - k - 1, \quad i = 0, 1, \ldots, n - 1,
\end{cases}
\]

are satisfied at \( t = 0 \) when \( I = [0, \infty) \), while the conditions

\[
\omega^{(i)}(+0) - \omega^{(i)}(-0) = \begin{cases} (-1)^{n-k}, & \text{if } i = n - k - 1, \\
0, & \text{if } i \neq n - k - 1, \quad i = 0, 1, \ldots, n - 1;
\end{cases}
\]

hold at \( t = 0 \) when \( I = (-\infty, \infty) \).

3) The function \( |\omega^{(n)}|^{1/(r-1)} \text{sgn} \omega^{(n)} \) has continuous derivatives up to order \( n - 1 \) on \( I \).

4) The function \( \omega \) satisfies the equation
\[
\frac{d^n}{dt^n} \left\{ \left| \omega^{(n)} \right| 1^{(r-1)} \text{sgn} \ \omega^{(n)} \right\} = (-1)^{n-1} c \left| \omega \right| 1^{(p-1)} \text{sgn} \ \omega,
\]

where \( c \) is a positive constant; here \( t \neq 0 \) when \( k = n - 1 \) and \( I = (-\infty, \infty) \).

Indeed, we can assert,

In order for (2.3) to be an extremal functional in problem (0.7) when \( N = \|\lambda\|_q \) and \( 1 < p, r < \infty \), it is necessary and sufficient that there exist a function \( \varphi \in B^p_0(N) \) such that \( \lambda = (-1)^n \varphi^{(n)} \) while the difference \( \omega = \theta - \varphi \in L_p^r \) have properties 1)–4); in this connection,

\[
e(\|\varphi^{(n)}\|_r) = \|\theta - \varphi\|_p.r.
\]

The necessity has just been proved. Let us prove the sufficiency. The functions \( \varphi \) and \( \psi = \theta \) satisfy the conditions of Lemma 1. It follows by virtue of the equalities \( x^{(k)}(0) = M^k_x \) and \( T = M_{\varphi} \) that (2.4) holds for any \( x \), and this implies the estimate

\[
e(\|\omega^{(n)}\|_r) \lesssim u(\theta, T) \lesssim \|\omega\|_p.r.
\]

Let us estimate \( e(N) \) from below. We put

\[y = (-1)^{n-1} c^{-1} \|\omega\|_{p.r}^{-1}(p-1) \|\omega^{(n)}\|_{1(r-1)} \text{sgn} \ \omega^{(n)}.\]

By virtue of properties 3) and 4), we have \( y \in W^p_{p,r} \) and

\[y^{(n)} = \|\omega\|_{p.r}^{-1}(p-1) \|\omega^{(n)}\|_{1(r-1)} \text{sgn} \ \omega,
\]

consequently \( y \in Q^n_{p,r} \).

Further, it is not difficult to verify that \( Ty = \|T\|\|y\|_r = \|\omega^{(n)}\|_{r} \|y\|_r \). Hence (see [10])

\[
e(\|\omega^{(n)}\|_r) \geq y^{(k)}(0) - \|\omega^{(n)}\|_r \|y\|_r = y^{(k)}(0) - Ty,
\]

and since

\[y^{(k)}(0) - Ty = \int \omega(t) y^{(n)}(t) dt = \|\omega\|_{p,r},
\]

we have

\[
e(\|\omega^{(n)}\|_r) \geq \|\omega\|_{p,r}.
\]

The assertion is proved.

It is evident that (2.8) is Euler’s equation for problem (2.2) (and hence for problem (0.7)).

We cite another corollary of Lemmas 1 and 2.

**Corollary 2.** If \( q = \infty \), then

\[E(N) = e(N).
\]

A proof is given in [11] for \( 1 \leq r < \infty \), and in [18] for \( k = n - 1, p = 1 \). We will therefore prove it here for \( r = \infty, 1 \leq p \leq \infty, 0 \leq k < n \), excluding the case \( k = n - 1, p = 1 \).
The quantity \(e(N)\) is finite \([11]\), and there exists an extremal functional \(T\) \([11], [16]\). It can be assumed by virtue of Lemma 2 that it has the form (1.5) on the set \(C (r = \infty)\). We define an operator \(T_C\) from \(C\) into \(C\) by the formula
\[
(T_Cx)(\tau) = Tx, \quad x_\tau(t) = x(t + \tau).
\]
Clearly, \(\|T_C\|^C_C = \|T\|^{C*} \leq N\). Moreover, for any function \(x \in W_{p,r}^m\) we have
\[
|x^{(k)}(\tau) - (T_Cx)(\tau)| = |x^{(k)}_\tau(0) - Tx_\tau| \leq e(N)\|x_\tau\|_{p'}.
\]
Hence
\[
E(N) \leq e(N).
\]
The reverse inequality can be obtained in the same way as was done in \([11]\) for \(1 \leq r < \infty\). Relation (2.10) also follows from the last inequality, (0.4) and (0.8). The corollary is proved.

**Theorem 1.** If \(m = n - k, q = \infty, q' = 1, 1 \leq p, r \leq \infty, p' = p/(p - 1), r' = r/(r - 1)\), then
\[
G(N) = F(N) = F(\theta, N) = e(N) \quad \text{for} \quad I = (-\infty, \infty),
\]
\[
g(N) = f(N) = f(\theta, N) = e(N) \quad \text{for} \quad I = [0, \infty).
\]

**Proof.** The assertion is valid for \(r = \infty, k = 0\). Further, it suffices by virtue of (2.2) and the inequalities \(F(\theta, N) \leq f(N) \leq G(N)\) to prove the inequality
\[
G(N) \leq e(N). \tag{2.11}
\]
We will first exclude the case \(k = n - 1\) and \(p = 1\).

Let \(T\) be an extremal functional of problem (0.7). If \(1 \leq r < \infty\), it has the form (2.3). If, on the other hand, \(r = \infty\), it can be assumed by virtue of Lemma 2 (with \(\psi = \theta\)) that it has the form (1.5) on \(C\).

According to Lemma 2, there exists a function \(\varphi_0 \in W_{p,r}^m\) having properties (1.7) and such that for any function \(x \in W_{p,r}^m\)
\[
x^{(k)}(0) - Tx = \int_0^\infty \rho(t)x^{(n)}(t)\,dt. \tag{2.12}
\]
(cf. (2.4)) when \(1 < p \leq \infty\), while
\[
x^{(k)}(0) - Tx = \int_0^\infty \rho(t)\,dx^{(n-1)}(t)
\]
when \(p = 1\), where
\[
\rho = \theta - \varphi_0, \tag{2.13}
\]
the function \(\theta\) being defined by relation (2.1). In this connection (cf. (2.5)),
\[
\|\rho\|_{p'} = \|\theta - \varphi_0\|_{p'} = e(N). \tag{2.14}
\]
For an arbitrary function \(\psi \in W_{1}^{n-k}\) we put \(\mu = (-1)^{n-k}\psi^{(n-k-1)}\) and define a functional \(T_\psi\) on \(L_r\) (or \(C\) if \(r = \infty\)) by the relation
\[ T_\varphi x = \int T_{\tau} \, d\mu(\tau), \quad x_\tau(t) = x(t + \tau); \quad (2.15) \]

clearly, \( T_\varphi \in L^*_r \) and

\[ \| T_\varphi \|_{L^*_r} \leq \| T \|_{L^*_r} \vee \mu \leq N \vee \psi^{(n-k-1)}. \quad (2.16) \]

On the basis of definition (1.1) we can write

\[ \int (x^{(k)}(\tau) - T_\tau x) \, d\mu(\tau) = M^k_\varphi x - T_\varphi x. \quad (2.17) \]

Suppose now \( 1 < p \leq \infty \). By virtue of (2.12) we have

\[ x^{(k)}(\tau) - T_\tau x = \int_0^t \rho(t) x^{(n)}(t + \tau) \, dt. \]

for \( x \in W^n_{p,r} \) and any \( \tau \in I \). Using this representation, we obtain from (2.17) the identity

\[ M^k_\varphi x - T_\varphi x = \int x^{(n)}(t) R(t) \, dt, \quad (2.18) \]

in which

\[ R(t) = \int_0^t \rho(t - \tau) \, d\mu(\tau), \quad \text{if} \quad I = [0, \infty). \quad (2.19) \]

\[ R(t) = \int_{-\infty}^t \rho(t - \tau) \, d\mu(\tau), \quad \text{if} \quad I = (-\infty, \infty), \]

Analogously, if \( p = 1 \), we have

\[ M^k_\varphi x - T_\varphi x = \int R(t) \, dx^{(n-1)}(t), \quad x \in W^n_{p,r}, \]

where the function \( R \) is also given by (2.19).

We recall that \( \mu = (-1)^{n-k} \psi^{(n-k-1)} \). Let

\[ S\psi = \psi - R; \quad (2.20) \]

clearly, \( S \) is a linear operator on the set \( W^n_{p,r} \). We will promptly prove that \( S \) is an operator of class \( \mathfrak{M}(N) \) or \( \mathfrak{M}^r(N) \) and that

\[ J(S) \leq e(N). \quad (2.21) \]

This will imply (2.11) and, consequently, the equalities

\[ e(N) = G(N) = J(S). \]

And this means, in particular, that the operator \( S \) determines a best linear method of approximation of a class by a class.

Applying Minkowski's inequality for integrals, we have

\[ \| R \|_{p'} \leq \| \rho \|_{p'} \vee \mu. \quad (2.22) \]
It follows by virtue of (2.20), (2.19) and (2.14) that 
\[ \| \psi - S\psi \|_{p'} = \| R \|_{p'} \leq e(N) \lor \psi^{(n-k-1)}, \]
and hence that (2.21) is valid.

Let us prove that if \( \psi \in \mathcal{W}_1^{n-k} \), then \( S\psi \in \mathcal{W}_p^n \) and
\[ \| (S\psi)^{(n)} \|_r \leq N \lor \psi^{(n-k-1)}. \] (2.23)

This assertion can be verified directly by differentiating the right side of (2.20). We proceed differently, however. From (2.18) and (2.22) we have for \( u(\psi, T_\psi) \) (see (1.3)) the estimate
\[ u(\psi, T_\psi) \leq e(N) \lor \mu < \infty. \]

According to Lemma 2, for a function \( \psi \) and a functional \( T_\psi \) there exists a function \( \varphi \in \mathcal{W}_p^n \) such that
\[ \| \psi^{(n)} \|_r \leq \| T_\psi \|_{L_r^n} \] (2.24)
and (1.9) holds on the class \( \mathcal{W}_p^n \). In the present case \( T_0 = T_\psi \), so that from (2.24) and (1.9) we have for infinitely differentiable functions \( x \) with compact support the relation
\[ \int [\psi(t) - \varphi(t) - R(t)] x^{(n)}(t) \, dt = 0. \]

It follows that
\[ S\psi = \psi - R = \varphi + P, \]
where \( P \) identically vanishes if \( I = [0, \infty) \), and is a polynomial of degree \( n-1 \) if \( I = (-\infty, \infty) \). But this, as is easily seen, implies \( S\psi \in \mathcal{W}_p^n \), and (2.23) holds by virtue of (2.16) and (2.24).

Thus, with the exception of the case \( k = n-1, p = 1 \), Theorem 1 is proved.

Suppose now \( p = 1 \) and \( k = n-1 \). If \( I = (-\infty, \infty) \), then \( \mathcal{W}_1^n = \mathcal{W}_1^1 \) is a class of functions of bounded variation. Such functions have a limit for \( t \to \pm \infty \). We define an operator \( S \) by putting
\[ S\psi = \frac{1}{2} \{ \psi(+\infty) + \psi(-\infty) \}. \] (2.25)

We have
\[ \psi(t) - (S\psi)(t) = \frac{1}{2} [\psi(t) - \psi(+\infty)] + \frac{1}{2} [\psi(t) - \psi(-\infty)], \]
which implies \( \| \psi - S\psi \|_{L_\infty} \leq \frac{1}{2} \lor \psi \). Consequently,
\[ G(N) \leq \| (S) \| \leq \frac{1}{2}. \] (2.26)

When \( I = [0, \infty) \) we put
\[ S\psi = 0, \quad \psi \in \mathcal{W}_1^1. \] (2.27)
Since \( \psi(0) = 0 \), it follows that \( |\psi(t)| = |\psi(t) - \psi(0)| \leq \sqrt{\psi} \), whence \( \| \psi - S\psi \|_{L^\infty} \leq 1 \) and
\[
g(N) \leq j(0) \leq 1. \tag{2.28}
\]

It is known [18] that when \( p = 1 \) and \( k = n - 1 \) (\( r < \infty \) if \( k = 0 \))
\[
e(N) = \sup_{x \in Q} \varphi^{(n-1)}(0) = u(\theta, 0) = \begin{cases} \frac{1}{2} & \text{for } I = (-\infty, \infty), \\ 1 & \text{for } I = [0, \infty). \end{cases} \tag{2.29}
\]

Inequality (2.11) follows from (2.26), (2.28) and (2.29) for the case under consideration. Thus Theorem 1 is completely proved.

The operator \( S \) defined on \( \mathcal{B}^{n-k}_{1} \) by (2.27), (2.25) and (2.20) yields a best linear method of approximation of the class \( \mathcal{B}^{n-k}_{1} \) by the class \( \mathcal{B}^{n}_{1}(N) \). Let us make some remarks concerning (2.20).

Suppose first \( I = [0, \infty) \). For \( \psi \in \mathcal{B}^{n-k}_{1} \), integrating by parts \( n - k - 1 \) times, we get
\[
\int_{0}^{t} \theta(t - \tau) d\psi^{(n-k-1)}(\tau) = \psi(t).
\]

Therefore (2.20) takes the form
\[
(S\psi)(t) = (-1)^{n-k} \int_{0}^{t} \varphi_{0}(t - \tau) d\psi^{(n-k-1)}(\tau). \tag{2.30}
\]

Here \( \varphi_{0} \) is the function of class \( \mathcal{B}^{n}_{1}(N) \) connected with an extremal operator \( T \) of problem (0.7) by relations (1.7), (1.4), (1.5), (2.13) and (2.14). If we now associate with a function \( \psi \in \mathcal{B}^{n-k}_{1} \) the function \( X \in \mathcal{B}^{n}_{1} \) such that \( X^{(k)} = \psi \), then, by way of illustration, (2.30) can also be written in the form
\[
S\psi = (-1)^{k} \int_{0}^{t} \lambda(t - \tau) X(\tau) d\tau \tag{2.31}
\]
for \( 1 \leq r < \infty \).

If \( I = (-\infty, \infty) \), (2.20) has the form
\[
(S\psi)(t) = \psi(t) - \int_{-\infty}^{t} [\theta(t - \tau) - \varphi_{0}(t - \tau)] d\psi^{(n-k-1)}(\tau). \tag{2.32}
\]

Suppose the function \( \omega = \theta - \varphi_{0} \) and its derivatives up to order \( n - 1 \) tend sufficiently rapidly to zero at infinity; more precisely, suppose \( \omega^{(i)}(t) = O(|t|^{-i}) \) for \( t \to \pm \infty \) (when \( i = 0, 1, \ldots, n - 1 \)). Then
\[
(S\psi)(t) = TX_{t}, \tag{2.33}
\]
where \( X_{t}(\tau) = X(t + \tau) \) and the function \( X \) satisfies the condition \( X^{(k)} = \psi \). In fact, since \( X^{(n-1)} \) is a function of bounded variation, we have \( X^{(n-1-i)}(t) = O(|t|^{i}) \) for \( t \to \pm \infty \) when \( i = 0, 1, \ldots, n - 1 \); and integrating the integrals
\[
\int_{t}^{\infty} \omega(t - \tau) d\psi^{(n-k-1)}(\tau), \quad \int_{-\infty}^{t} \omega(t - \tau) d\psi^{(n-k-1)}(\tau)
\]
by parts \( n \) times, we get (2.33) from (2.32).
Taikov [14] solved problem (0.7) for \( I = (-\infty, \infty) \) and \( p = r = 2 \). In the present case it has turned out that the function \( \omega \) (and its derivatives) satisfy the relation
\[
\omega(t) = O(e^{-\mu|t|}) \quad \text{for} \quad t \to \pm \infty.
\]
Consequently, here the best linear method of approximation (of the class \( W^{n-k}_1 \) by the class \( W^m_2 \) in the metric of \( L_2 \)) can be written in the form (2.33).

Theorem 1 implies

**Corollary 3.** When \( I = (-\infty, \infty) \), the inequality
\[
\|\varphi - \varphi^{(n)}\|_{L^p} \geq K \beta^\alpha x^a,
\]
holds for any function \( \varphi \in W^m_2 \); when \( I = [0, \infty) \), (2.34) holds for any function \( \varphi \in W^m_2 \).  Here the function \( \theta \) is defined by (2.1), \( K \) is the least constant in (0.1), \( q = \infty \) and conditions (0.23) are satisfied. Inequality (2.34) is sharp.

In fact, according to Theorem 1 and (0.4) we have
\[
\mathcal{F}(\theta, N) = \inf_{\varphi \in \mathcal{W}^m_2(N)} \|\theta - \varphi\|_{L^p} \geq \Omega(N),
\]
from which (2.34) readily follows. The sharpness of (2.34) is a consequence of (0.8).

As was noted by Steklin, this inequality was essentially proved (in another way) for \( p = r = 2 \) and \( I = (-\infty, \infty) \) by Taikov [14].

§3. Connection of inequality (0.1) with the approximation of a class by a class

In the present section we will essentially make use of Lemma 1 of [16], which we cite in the following form.

**Lemma 3** (V. N. Gabushin [16]). Suppose \( X \) is a Banach space, \( A \) is a (not necessarily bounded) linear functional on \( X \), and \( Q \) is a symmetric convex set belonging to the domain of definition of \( A \). Then the equality
\[
\inf_{\|x\|_{X^*} \leq N} \sup_{x \in Q} (Ax - Tx) = \sup_{x \in Q} (Ax - N \|x\|_X)
\]
holds for any \( N > 0 \), and there exists an extremal functional.

We call a function \( \psi \) maximal in problems (0.10), (0.15) if, respectively,
\[
F(\psi, N) = F(N), \quad \psi \in B^m_{\psi},
\]
\[
j(\psi, N) = j(N), \quad \psi \in b^m_{\psi}.
\]

**Theorem 2.** Suppose \( 1 \leq p, q, r \leq \infty \),
\[
m = n - k, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad \frac{1}{q} + \frac{1}{q'} = 1, \quad \frac{1}{r} + \frac{1}{r'} = 1
\]
and condition (0.27) is satisfied. Then
\[
\beta^{\alpha/\beta} K^{1/\beta} N^{-x/\beta} = \begin{cases} F(N) & \text{for} \quad I = (-\infty, \infty), \\ j(N) & \text{for} \quad I = [0, \infty). \end{cases}
\]

In addition, the function \( \theta \) (see (2.1)) is maximal in problems (0.10), (0.15) for \( q = \infty \).
If there exists an extremal function \( z \) in (0.1) for \( 1 \leq q < \infty \) and it is normalized by the conditions
\[
\| z^{(n)} \|_p = 1, \quad \| z \|_r = (\alpha K)^{1/\beta} N^{-1/\beta},
\] (3.3)
then the function \( \eta \) for which
\[
\eta^{(n-k)} = \| z^{(k)} \|_{q^*} \| z^{(k)} \|_{r^*}^{-1} \text{ sgn } z^{(k)}
\]
will be maximal.

We note that conditions (0.2) and (0.14) coincide, so that if (0.2) is violated, both of the quantities in (3.2) are infinite. If, on the other hand, \( k = 0 \) and \( q = r \), the left side of (3.2) is degenerate (\( \beta = 0 \)).

**Proof.** The case \( q = \infty, p = 1, k = n - 1 \) is excluded from the proof presented below. When \( q = \infty \), however, all of the assertions of the theorem follow from Theorem 1 and (0.8).

Consider the quantity
\[
\sigma (N) = \sup_{\psi \in \mathcal{S}^{n-k}_{q^*}} \sigma (\psi, N),
\]
where \( \sigma (\psi, N) \) is defined by (1.14), (1.3) and (1.1). By virtue of (1.15), we have
\[
\sigma (N) = \sup_{\psi \in \mathcal{S}^{n-k}_{q^*}} \mathcal{F} (\psi, N) = \mathcal{F} (N).
\]
On the other hand, according to Lemma 3,
\[
\sigma (\psi, N) = \sup_{x \in Q_{p^*}} \{ M^k_x - N \| x \|_r \},
\] (3.4)
which implies
\[
\sigma (N) = \sup_{x \in Q} \sup_{\psi \in \mathcal{S}^{n-k}_{q^*}} (M^k_x - N \| x \|_r).
\]
And since
\[
\sup_{\psi \in \mathcal{S}^{n-k}_{q^*}} M^k_x = \| x^{(k)} \|_{q^*},
\] (3.5)
we finally obtain
\[
\sigma (N) = \sup_{x \in Q} (\| x^{(k)} \|_q - N \| x \|_r) = \Omega (N).
\]
Relation (3.2) is thereby proved.

Let us prove the second part of the theorem. If \( q = \infty \), we have
\[
\Omega (N) = \sup_{x \in Q} (\| x^{(k)} \|_\infty - N \| x \|_r)
\]
\[
= \sup_{x \in Q} (x^{(k)} (0) - N \| x \|_r) = \sup_{x \in Q} (M^k_x - N \| x \|_r),
\]
from which, with the use of (3.4) for \( \psi = \theta \) and (1.15), we get
\[
\Omega (N) = \sigma (\theta, N) = \mathcal{F} (\theta, N),
\]
and since \( \Omega (N) = F(N) \), the function \( \theta \) is maximal.
Suppose \( 1 \leq q < \infty \). We will show that under the conditions of the theorem
\[
\Omega (N) = \sigma (\eta, N) = \mathcal{F} (\eta, N).
\] (3.6)

Clearly \( \| \eta^{(n-k)} \|_p = 1 \), and \( M_\eta^k x \leq \| x^{(k)} \|_q \), so that
\[
\sigma (\eta, N) \leq \Omega (N).
\]

On the other hand,
\[
\sigma (\eta, N) \geq M_\eta^k z - N \| z \|_r = \| z^{(k)} \|_q - N \| z \|_r.
\]

The function \( z \) is an extremal in (0.1) and satisfies (3.3); consequently
\[
\| z^{(k)} \|_q = K \| z \|_p^a = \alpha^{a/\beta} K^{1/\beta} N^{-a/\beta},
\]
from which we get
\[
\sigma (\eta, N) \geq \beta \alpha^{a/\beta} K^{1/\beta} N^{-a/\beta} = \Omega (N).
\]

Thus (3.6) holds. Theorem 2 is completely proved.

Remark 2. If, at least when \( 1 < q' \leq \infty \) \( (1 \leq q < \infty) \), a set \( \mathcal{B} \subset \mathcal{B}_q^{n-k} \) is such that
\[
\sup_{\xi \in \mathcal{B}_q} \int \psi^{(n-k)} \xi \, dt = \| \xi \|_q,
\]
for any function \( \xi \in L_q \), then
\[
\sup_{\psi \in \mathcal{D}_q} \mathcal{F} (\psi, N) = F(N) = \Omega (N).
\]

This assertion readily follows from the proof of (3.2).

We note in addition that a maximal function is generally not unique. In fact, suppose \( q = \infty \), let \( z \) be an extremal function in (0.1) and let \( t_i \) be the points at which
\[
| z^{(k)} (t_i) | = \| z^{(k)} \|_C.
\] (3.7)

It will be assumed that \( z \) satisfies (3.3). We choose numbers \( a_i \) so that
\[
\sum | a_i | = 1, \quad a_i z^{(k)} (t_i) \geq 0,
\]
and define a function \( \eta \) by the equality \( \eta(t) = \sum a_i \theta (t - t_i) \). Then
\[
\eta \leq \eta^{(n-k)} \frac{\eta^{(n-k-1)}}{1}, \quad \eta^{(n-k-1)} = 1 \quad \text{and} \quad \int z^{(k)} (t) \, d\eta^{(n-k-1)} (t) = \| z^{(k)} \|_C.
\]

It follows in the same way as in the proof of Theorem 2 that \( \eta \) satisfies (3.6), i.e. \( \eta \) is maximal. It is evident that such a function is not unique if there exist several points \( t_i \) satisfying (3.7). From the results of [5] we see that this is the case, for example, when \( p = q = r = \infty \) and \( I = (-\infty, \infty) \).

§4. The approximation of a differentiation operator

and the linear problem of approximation of a class by a class

A connection analogous to (3.2) exists between \( E(N) \) and the quantities \( G(N) \) and \( g(N) \).
THEOREM 3. If $1 < q \leq \infty$, $1 < p \leq \infty$, $1 \leq r \leq \infty$, and relations (3.1) are satisfied, then
\begin{equation}
E (N) = G (N) \quad \text{for} \quad I = (-\infty, \infty),
\end{equation}
\begin{equation}
E (N) = g (N) \quad \text{for} \quad I = [0, \infty).
\end{equation}

PROOF. Since these assertions are contained in Theorem 1 when $q = \infty$, it will be assumed that $1 < q < \infty$.

Suppose $E(N) < \infty$ and suppose $U(T)$ is finite for some bounded linear operator $T$ from $L_r$ into $L_q$ with $\|T\| \leq N$. With a function $\psi \in \mathcal{W}_{q}^{n-k}$ we associate a functional $T_\psi$ by the formula
\begin{equation}
T_\psi x = \int \psi^{(n-k)} T x \, dt;
\end{equation}
we have
\begin{equation}
\|T_\psi\|_{L_p} \leq \|\psi^{(n-k)}\|_{r'} N,
\end{equation}
\begin{equation}
\sup_{x \in \mathcal{X}} \left\{\int \psi^{(n-k)} x^{(k)} dt - T_\psi x\right\} \leq U(T) \|\psi^{(n-k)}\|_{r'}.
\end{equation}
According to Lemma 2, there exists a unique function $\varphi \in \mathcal{W}_{p}^{n}$ with the properties
\begin{equation}
\|\psi - \varphi\|_{p'} \leq U(T) \|\psi^{(n-k)}\|_{r'}, \quad \|\varphi^{(n)}\|_{r'} \leq N \|\psi^{(n-k)}\|_{r'},
\end{equation}
Thus $\varphi = S\psi$, where $S$ is a single-valued operator from $\mathcal{W}_{q}^{n-k}$ into $\mathcal{W}_{p}^{n}$; in this connection, $S$ satisfies condition (0.9) and
\begin{equation}
J (S) \leq U(T).
\end{equation}
Let us show that the operator $S$ is linear. For any functions $\psi_1, \psi_2 \in \mathcal{W}_{q}^{n-k}$ we have
\begin{equation}
T_{\psi_1 + \psi_2} = T_{\psi_1} + T_{\psi_2}.
\end{equation}
It follows by virtue of (1.7) that
\begin{equation}
(S' (\psi_1 + \psi_2))^{(n)} = (S\psi_1)^{(n)} + (S\psi_2)^{(n)}
\end{equation}
for $r < \infty$, while
\begin{equation}
(S (\psi_1 + \psi_2))^{(n-1)} = (S\psi_1)^{(n-1)} + (S\psi_2)^{(n-1)} + c
\end{equation}
for $r = \infty$, where $c$ is a certain constant. Hence
\begin{equation}
S (\psi_1 + \psi_2) = S\psi_1 + S\psi_2 + P,
\end{equation}
where $P$ is a certain polynomial of degree $n - 1$. According to the first of inequalities (4.3) we have $P \in L_{p'}$, and since $p' < \infty$, we get $P \equiv 0$, i.e. the operator $S$ is linear.

It now follows from (4.4) that
\begin{equation}
G (N) \leq E (N).
\end{equation}

Suppose $G(N) < \infty$, and suppose an operator $S$ of class $\mathcal{M}(N)$ or $\mathcal{M}'(N)$ has the property $J(S) < \infty$. We will first consider the case $r' > 1$ ($r < \infty$). It is not difficult to see that the value $(S\psi)^{(n)}$ is uniquely determined by the derivative $\psi^{(n-k)}$, so that
\[(S\psi)^{(n)} = T^*\psi^{(n-k)}.\] Clearly, \(T^*\) is a linear operator from \(L_{q'}\) into \(L_r\), and \(\|T^*\| \leq N.\)

Let \(T\) denote the adjoint of \(T^*\) when \(1 < r' < \infty\), and the restriction to \(L_r\) of the adjoint of \(T^*\) when \(r' = \infty\). Clearly, \(\|T\|_{L_q} \leq N;\) also, for any functions \(\psi \in \mathcal{U}_{q-r}^a\) and \(x \in L_r\)

\[
\int x (S\psi)^{(n)} dt = \int \psi^{(n-k)} Tx dt.
\]

Applying Lemma 1 to the functions \(\psi\) and \(\varphi = S\psi\), we obtain

\[
\int \psi^{(n-k)} \{x^{(k)} - Tx\} dt = \int \psi^{(n-k)} x^{(k)} dt - \int \psi^{(n-k)} Tx dt \leq \|x^{(k)}\|_{L_p} \|\psi\| - S\psi\|_{L_p}.
\]

Hence \(U(T) \leq J(S)\); therefore \(E(N) \leq J(S)\), and consequently

\[
E(N) \leq G(N). \tag{4.6}
\]

When \(r' = 1\), the value \((S\psi)^{(n-1)} = \lim_{t \to \infty} (S\psi)^{(n-1)}(t)\) is uniquely determined by the function \(\psi^{(n-k)}\), and we arrive at an operator \(T^*\) from \(L_{q'}\) into the space \(V\) of functions of bounded variation. By considering the restriction to \(C\) of the adjoint of \(T^*\), we obtain (4.6) in the same way as when \(r' > 1\).

Thus, if either \(E(N)\) or \(G(N)\) is finite, so the other is and they are equal. Theorem 3 is proved.

**Remark 3.** In the proof of the latter theorem (at least when \(1 < q < \infty, 1 \leq r < \infty\)) a one-to-one correspondence between the sets of operators \(S\) and \(T\) for which the corresponding deviations \(J(S)\) and \(U(T)\) are finite is established by the formula

\[
(-1)^k (S\psi)^{(n)} = T^*\psi^{(n-k)}, \tag{4.7}
\]

where \(T^*\) is the adjoint of the operator \(T\); in this connection, \(U(T) = J(S)\). It follows that if either \(S\) or \(T\) is an extremal operator, so is the other (in the appropriate problem).

**§ 5. The independence of the approximation of a class by a class from the set \(I\)**

Ju. N. Subbotin [24] obtained one and the same estimate from below of the quantities \(F(N, [0, \infty))\) and \(F(N, (-\infty, \infty))\) in the cases \(p' = q' = r' = \infty\) and \(p' = r' = q' = r = 1\), and also proved that this estimate is sharp for \(n \leq 5\). In [25] there is obtained an upper estimate of these quantities for arbitrary \(n \geq 2\) in the case \(p' = q' = r' = \infty\) that coincides with an upper estimate in [24]. Consequently, in these cases \(F(N)\) does not depend on \(I\). This fact holds for arbitrary values of the parameters.

**Theorem 4.** The following equalities are valid:

\[
F(N, [0, \infty)) = F(N, (-\infty, \infty)), \tag{5.1}
\]

\[
f(N, [0, \infty)) = f(N, (-\infty, \infty)). \tag{5.2}
\]

**Proof.** Any function \(y \in B^L_1(\mu, [0, \infty))\) can be extended by a polynomial of degree \(l - 1\) on the half-line \((-\infty, 0]\) to a function of class \(B^L_1(\mu, (-\infty, \infty))\). Conversely,
the restriction of an arbitrary function \( y \in B^I_1(\mu, (-\infty, \infty)) \) to the half-line \([0, \infty)\) is a function of class \(B^I_1(\mu, [0, \infty))\). Therefore

\[
F(N, [0, \infty)) = \sup_{\Psi \in B^I_1(1, (-\infty, \infty))} \inf_{\Phi \in B^I_1(N, (-\infty, \infty))} \| \Psi - \Phi \|_{L^p[0, \infty)}
\]

and consequently

\[
F(N, [0, \infty)) \leq F(N, (-\infty, \infty)).
\]

In exactly the same way,

\[
f(N, [0, \infty)) \leq f(N, (-\infty, \infty)).
\]

Let us prove the reverse inequalities. It will be assumed in this connection that condition (0.14) is satisfied, since otherwise \(F(N) = \infty\) and hence \(f(N) = \infty\) inasmuch as \(f(N) \geq F(N)\).

We begin with the inequality

\[
f(N, (-\infty, \infty)) \leq f(N, [0, \infty)), \quad (5.3)
\]

Let \( \psi \in b^{q''}_{m}(1, (-\infty, \infty)) \), and let \( \psi_1 \) and \( \psi_2 \) denote the functions defined on \([0, \infty)\) by the formulas \( \psi_1(t) = \psi(t) \) and \( \psi_2(t) = \psi(-t) \). We put \( \epsilon_i = \| \psi_i^{(m)} \|_{q'} \) \((i = 1, 2)\) and assume that \( q'^{p'} = 0 \) if \( p' = \infty \) \((1 \leq q' < \infty)\). Then for the quantities

\[
f(\psi, N) = f(\psi, N, (-\infty, \infty)) \quad \text{and} \quad f(\psi_i, N \epsilon_i^{q''} r') = f(\psi_i, N \epsilon_i^{q''} r', [0, \infty))
\]

we have

\[
f(\psi, N) \leq \left\{ \sum f_p'(\psi_i, N \epsilon_i^{q''} r') \right\}^{1/p'} \quad (5.4)
\]

when \( 1 \leq p' < \infty \), and

\[
f(\psi, N) \leq \max f(\psi_i, N \epsilon_i^{q''} r') \quad (5.5)
\]

when \( p' = \infty \). But

\[
f(\psi_i, N \epsilon_i^{q''} r') = \epsilon_i f(\psi_i \epsilon_i^{-1}, N \epsilon_i^{q''} r' - 1, [0, \infty)) \leq \epsilon_i f(N \epsilon_i^{q''} r' - 1, [0, \infty)),
\]

which by virtue of the equality \( f(N) = f(1)N^{-v} \) implies

\[
f(\psi_i, N \epsilon_i^{q''} r') \leq \epsilon_i f(N, [0, \infty)), \quad (5.6)
\]

where

\[
v = \left( m - \frac{1}{q'} + \frac{1}{p'} \right) / \left( n - m + \frac{1}{q'} - \frac{1}{r'} \right), \quad \gamma = 1 + v - \frac{v q'}{r'}.
\]

Suppose \( 1 \leq p' < \infty \). Then \( q' < \infty \), since, according to (0.14), \( p' = r' = \infty \) if \( q' = \infty \). From (5.4) and (5.6) we get

\[
f(\psi, N) \leq f(N, [0, \infty)) \left( \epsilon_1^{p''} + \epsilon_2^{p''} \right)^{1/p'}.
\]

Since
\[
\frac{p'\gamma}{q'} - 1 = \frac{1}{p'} \left( n - \frac{n - m}{p'} - \frac{m}{r'} \right) \left( n - m + \frac{1}{q'} - \frac{1}{r'} \right),
\]
we have \( p'\gamma \geq q' \) by virtue of (0.14). Also, \( 0 \leq \varepsilon_i \leq 1 \). Therefore \( \varepsilon_2^p \gamma + \varepsilon_2'^p \gamma \leq \xi_1^q' + \xi_2^q' \leq 1 \). Hence
\[
f(\psi, N, (\infty, \infty)) \leq f(N, [0, \infty)),
\]
and this leads to (5.3).

If \( p' = \infty \) and \( 1 \leq q' < \infty \), then
\[
\gamma = q' \left( \frac{n}{q'} - \frac{m}{r'} \right) \left( n - m + \frac{1}{q'} - \frac{1}{r'} \right),
\]
which implies \( \gamma \geq 0 \) according to (0.14); analogously, we also have \( \gamma \geq 0 \) if \( p' = q' = \infty \) (\( r' = \infty \)). Inequality (5.7) (and hence (5.3)) for \( p' = \infty \) now follows from (5.6) and (5.5).

Thus (5.2) is proved.

Let us establish the inequality
\[
F(N, (\infty, \infty)) \leq F(N, [0, \infty)).
\]
It suffices to prove that if
\[
\bar{N} \geq N, \quad A \geq F(N, [0, \infty)),
\]
then
\[
F(\bar{N}, (\infty, \infty)) \leq A,
\]
since this implies \( F(\bar{N}, (\infty, \infty)) \leq F(N, [0, \infty)) \), from which (5.8) follows by virtue of (0.13).

Let \( \bar{B} = B_q^m(1, (\infty, \infty)) \) denote the set of functions \( \psi \in B_q^m(1, (\infty, \infty)) \) satisfying the condition \( \psi(t) = 0 \) for \( t \leq 0 \), and let us prove the equality
\[
F(N, (\infty, \infty)) = \sup_{\psi \in \bar{B}} F(\psi, N, (\infty, \infty)).
\]
which follows from Theorem 1 when \( q' = 1 \) since \( \theta \in \bar{B} \).

Suppose \( 1 < q' \leq \infty \). For any function \( \zeta \in L_q \left( \frac{1}{q} + \frac{1}{q'} = 1 \right) \), we have
\[
\|\zeta\|_q = \sup_{\|\zeta\|_{\infty} \leq 1} \int_{\frac{1}{q}} \zeta dt.
\]
Here it is possible to confine oneself to the set of functions with compact support, so that in (3.5) it is possible to confine oneself to the functions \( \psi \in B_q^m \) whose derivatives of order \( m \) vanish for \( t < a \) (the number \( a \) depends on \( \psi \)). The class \( B_\theta \) of such functions satisfies the conditions of Remark 2 to Theorem 2. But this implies (5.11).

Suppose \( \psi \in \bar{B} \). For an arbitrary number \( \delta > 1 \) we put \( \psi_\delta(t) = \psi(t - \delta) \). The restriction of \( \psi_\delta \) to \( [0, \infty) \) is a function of class \( B_q^m(1, [0, \infty)) \). It will be assumed in the sequel that \( \bar{N} \) and \( A \) are two fixed numbers satisfying (5.9). For each \( \delta \) we choose a function \( \varphi_\delta \in B_q^m(N, [0, \infty)) \) so that
\[ F(\psi, N, (0, \infty)) \leq \|\psi - \varphi\|_{L^p(0, \infty)} \leq A. \]

Let \( \kappa \) be an infinitely differentiable function on \((-\infty, \infty)\) such that \( 0 \leq \kappa \leq 1 \), \( \kappa(t) = 0 \) for \( t \leq 0 \) and \( \kappa(t) = 1 \) for \( t \geq 1 \), and let \( y_\delta(t) = \varphi_\delta(t) \kappa(t\delta^{-1}) \). Since \( \psi_\delta(t) = 0 \) for \( 0 \leq t \leq \delta \), we have

\[ \psi_\delta(t) - y_\delta(t) = \kappa(t\delta^{-1}) \{\psi_\delta(t) - \varphi_\delta(t)\}, \quad t \geq 0, \]

and therefore

\[ \|\psi_\delta - y_\delta\|_{L^{r'}} \leq \|\psi_\delta - \psi_\delta\|_{L^{r'}} \leq A. \]

When \( 1 < r' \leq \infty \) we have, according to Leibniz' formula,

\[ y_\delta^{(n)}(t) = \varphi_\delta^{(n)}(t)\kappa(t\delta^{-1}) + \rho(t), \]

\[ \rho(t) = \sum_{i=1}^{n} C_i^r \varphi_\delta^{(n-i)}(t) \delta^{-i} \kappa^{(i)}(t\delta^{-1}). \]

On the intervals \([0, \delta]\) the functions \( \varphi_\delta \) have the properties \( \|\varphi_\delta\|_{L^p} \leq A \) and \( \|\varphi_\delta^{(n)}\|_{L^r} \leq N \). It follows from Theorem 1 of [6] that when \( \delta \geq 1 \) there exists a constant \( c \) not depending on \( \delta \) (and hence not on \( \varphi_\delta \)) such that \( \|\varphi_\delta^{(n)}\|_{C[0, \delta]} \leq c \) for \( t = 0, 1, \ldots, n-1 \). Therefore \( \|\rho\|_{L^r} \leq c_1 \delta^{1/r'-1} \) (where \( c_1 \) does not depend on \( \delta \)). Consequently, for sufficiently large \( \delta \)

\[ \|y_\delta^{(n)}\|_{L^r([0, \infty))} = \|y_\delta^{(n)}\|_{L^r((-\infty, \infty))} \leq N. \]

Let \( y \) denote the function defined on \((-\infty, \infty)\) by the equality \( y(t) = y_\delta(t + \delta) \). We have \( \|y^{(n)}\|_{L^r} \leq N \) and

\[ \|\psi - y\|_{L^r((-\infty, \infty))} = \|\psi_\delta - y_\delta\|_{L^r(0, \infty)} \leq A, \]

which implies

\[ F(\psi, N, (-\infty, \infty)) \leq A. \quad (5.12) \]

Inequality (5.10) follows from this result by virtue of (5.11). Thus (5.8) and hence (5.1) hold when \( r' > 1 \).

Suppose \( r' = 1 \). The case \( n = 1 \) \((m = n = 1, r' = 1)\) is trivial (see the end of the Introduction); it will therefore be assumed that \( n \geq 2 \). Inasmuch as \( \sqrt{\delta} \varphi_\delta^{(n-1)} \leq N \), we have for some interval \( I_\delta = [a, b] \subset [0, \delta] \)

\[ b - a = V\delta, \quad \sqrt{\varphi_\delta^{(n-1)}} \leq 2N/\sqrt{\delta}. \quad (5.13) \]

Let us show that the function \( \zeta = \varphi_\delta^{(n-1)} \) cannot be large on this interval, viz. that there exists a constant \( c \) depending only on \( A \) and \( N \) such that

\[ |\varphi_\delta^{(n-1)}(t)| \leq cV\delta \quad \text{for} \quad t \in I_\delta = [a, b]. \quad (5.14) \]

Suppose this is not true. Then, by virtue of (5.13), the function \( \zeta \) will preserve its sign and satisfy the condition

\[ |\zeta(t)| \geq \frac{c - 2N}{V\delta} = \chi(\delta) \]
on $I_\delta$ for $c > 2N$. But then $|\varphi^{(n-2)}_\delta(t)| \geq \chi(\delta)$ on either the interval $[a, (a + b)/2 - 1]$, or the interval $[(a + b)/2 + 1, b]$. An analogous result can be obtained for $\varphi^{(n-1-\delta)}_\delta$ by induction on $i \leq n - 1$. In particular, there exists an interval $[\alpha, \beta] \subset [a, b]$ of length $2^{1-n}\sqrt{\delta} - 2$ on which $|\varphi_\delta(t)| \geq \chi(\delta)$. It readily follows that
\[
\|q_\delta\|_{L^p(I_\delta)} \geq \chi(\delta) \left\{ \int_0^{1/n} t^{n'} dt \right\}^{1/n'}.
\]
Thus, if $\delta \geq 1$,
\[
\|q_\delta\|_{L^p(I_\delta)} \geq c_1 \chi(\delta) \sqrt{\delta} = c_1 (c - 2N),
\]
where $c_1$ depends only on $p'$ and $n$. By assumption, the constant $c$ can be taken arbitrarily large. This contradicts the condition $\|q_\delta\|_{L^p([0, \delta])} \leq A$. Consequently a constant $c$ satisfying (5.14) exists.

Let $y_\delta(t) = \varphi_\delta(t) \chi((t - a)\delta^{-1/2})$. We then have
\[
y^{(n-1)}_\delta(t) = \varphi^{(n-1)}_\delta(t) \chi((t - a)\delta^{-1/2}) + \rho(t),
\]
\[
\rho(t) = \sum_{i=1}^{n-1} C^{(n-1-i)}_{n-1} \varphi^{(n-1-i)}_\delta(t) \delta^{-1/2} \chi^{(i)}((t - a)\delta^{-1/2}),
\]
which implies
\[
\sum_{-\infty}^{\infty} y^{(n-1)}_\delta = \sum_{-\infty}^{\infty} \chi \sup_{a \leq t \leq b} |y^{(n-1)}_\delta(t)| + \rho \|q_\alpha\|_{L^p(a, b)}.
\]
(5.15)

As before, it follows from Theorem 1 of [6] by virtue of (5.2) and the inequality $\|q_\delta\|_{L^p(a, b)} \leq A$ that there exists a constant $c_2$ for which $\|q_\delta \|_{C[a, b]} \leq c_2$, $i = 0, 1, \ldots, n - 2$. Therefore
\[
\|\rho\|_{L^p(a, b)} \leq c_3 \left( \|q_\delta\|_{L^p(a, b)} + \delta^{-1/2} \right).
\]
(5.16)

According to (5.14)--(5.16), for sufficiently large $\delta$
\[
\sum_{-\infty}^{\infty} y^{(n-1)}_\delta \leq N.
\]

In addition,
\[
\|q_\delta - y_\delta\|_{L^p(-\infty, \infty)} \leq \|q_\delta - y_\delta\|_{L^p(0, \infty)} \leq A.
\]

From these results, as in the case $r' > 1$, we obtain (5.12) and hence (5.8) and (5.1). Theorem 4 is proved.

Let us consider the dependence on $I$ of the quantities $G(N)$ and $g(N)$.

The following inequality holds:
\[
G(N, [0, \infty)) \leq G(N, (-\infty, \infty)).
\]
(5.17)

In fact, let $S \in \mathfrak{A}(N, (-\infty, \infty))$, and let $\bar{\psi} \in B^m_q(1, (-\infty, \infty))$ denote the function coinciding on $[0, \infty)$ with a given function $\psi \in B^m_q(1, [0, \infty))$ and being a polynomial on the half-line $(-\infty, 0]$. We define an operator $S^+$ by letting $S^+ \psi$ denote the restriction of the function $S\bar{\psi}$ to the half-line $[0, \infty)$. It is not difficult to verify that
\[ S^+ \in \mathcal{W}(N, [0, \infty)) \text{ and} \]
\[ \sup_{\psi} \| \psi - S^+ \|_{L^p(0, \infty)} \leq \sup_{\psi} \| \overline{\psi} - S \|_{L^p(0, \infty)}, \]

where \( \psi \in B^m_q(1, [0, \infty)) \) and \( \overline{\psi} \in B^m_q(1, (-\infty, \infty)). \) But this implies (5.17).

The inequality

\[ g(N, [0, \infty)) \leq g(N, (-\infty, \infty)) \]

(5.18)
can be obtained analogously to (5.17).

It will now be shown that the reverse inequality holds, and hence

*The quantity \( g(N) \) does not depend on \( I \), i.e.*

\[ g(N, [0, \infty)) = g(N, (-\infty, \infty)). \]

(5.19)

By virtue of Theorem 3, Remark 3 to Theorem 3, condition (0.6) for the finiteness of \( E(N) \) and inequality (5.18), it suffices to obtain the inequality

\[ g(N, [0, \infty)) \geq g(N, (-\infty, \infty)) \]

under the assumption \( r' \geq q', p' \geq q' \).

Let \( \overline{S} \in \mathcal{W}(N, [0, \infty)), \) and let \( S \) denote the operator defined on \( w^m_q \) for \( I = (-\infty, \infty) \) by the formula

\[ (S\psi)(t) = \begin{cases} (\overline{S}\psi_1)(t), & t \geq 0, \\ (\overline{S}\psi_2)(-t), & t \leq 0, \end{cases} \]

where \( \psi_1(t) = \psi(t), \psi_2(t) = \psi(-t) \) for \( t \in [0, \infty). \)

Clearly, \( S\psi \) has a sufficient number of derivatives and, together with its derivatives up to order \( n - 1 \), satisfies the null boundary condition at \( t = 0 \). In addition,

\[ \| (S\psi)^{(n)} \|_{L^{r'}(-\infty, \infty)} = \left\{ \sum \| (\overline{S}\psi_i)^{(n)} \|_{L^{r'}(0, \infty)} \right\}^{1/r'} \leq N \left( \| \psi_1^{(n)} \|_{L^{r'}} + \| \psi_2^{(n)} \|_{L^{r'}} \right)^{1/r'} \]

\[ = N \left\{ \sum \| \psi_i^{(n)} \|_{L^{r'}} \right\}^{1/r'}. \]

Hence, if \( r' \geq q' \),

\[ \| (S\psi)^{(n)} \|_{L^{r'}(-\infty, \infty)} \leq N \| \psi^{(m)} \|_{L^{q'}(-\infty, \infty)}, \]

i.e. \( S \in \mathcal{W}(N, (-\infty, \infty)). \)

Let us estimate the quantity

\[ \| \psi - S\psi \|_{r'} = \| \psi - S\psi \|_{L^{r'}(-\infty, \infty)}. \]

Clearly,

\[ \| \psi - S\psi \|_{r'} = \left\{ \sum \| \psi_i - \overline{S}\psi_i \|_{r'(0, \infty)} \right\}^{1/r'}. \]

And since

\[ \| \overline{\psi} - S\overline{\psi} \|_{r'} \leq j(S) \| \overline{\psi}^{(m)} \|_{r'} \]

for \( I = [0, \infty) \) and \( \overline{\psi} \in w^m_q \), we have, as before,
\[ \| \Psi - S\Psi \|_{p'} \leq j(S) \left( \sum \left( \| \Psi_i^{(m)} \|_l \right)^{p'/q'} \right)^{1/p'}, \]

which implies the inequality

\[ \| \Psi - S\Psi \|_{p'} \leq j(S) \| \Psi^{(m)} \|_{L_{q'}(-\infty, \infty)} \]

for \( p' \geq q' \). Consequently

\[ j(S, (-\infty, \infty)) \leq j(S, [0, \infty)). \]

The assertion is thereby proved.

**Remark 4.** When \( q = \infty \), it follows from Theorems 1--3 and relations (5.17) and (5.19) that the quantities \( F(N), G(N), f(N) \) and \( g(N) \) do not depend on \( l \) and satisfy the equalities \( F(N) = G(N) \) and \( f(N) = g(N) \).

**BIBLIOGRAPHY**


Translated by S. SMITH